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THE MATHEMATICS STUDENT

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PROBLEM OF TIME IN PHYSICS AND MATHEMATICS

BY

SUKUMAR RANJAN DAS, M.A., Ph.D.

Time has been regarded as an all-powerful agent conditioning the exterior world by the scientists whenever they had set upon pondering over its nature and functions. After a long period of philosophical speculations the real age of scientific thinking in Europe began with Leonardo da Vinci (1452-1519), who was a universal genius. He held that whatever could be measured could be known, what could not be measured could not be known and that the relations of motion could be reduced to mathematical formulae; hence all occurrences could be explained in terms of motion and its laws. These laws which form the basis of the study of mechanics and the measurement of motion, time and space were discovered and formulated by Leonardo da Vinci. More effective was the work of Nicholaus Copernicus (1473-1543) who reached his heliocentric theory rather by speculation than by observation, but it induced thoughtful people to abandon the geocentric theory of the universe. Just after a short time Tycho Brahe (1546-1601) made the most important advance in the accuracy and number of astronomical observations leading to the measurement of time. By way of compromise between the heliocentric and geocentric theories he suggested the possibility that the sun and the moon move round the earth, while the other planets move round the sun.

About this time Galileo Galilei (1564-1641) proved the correctness of the Copernican Theory by his telescopical observations and created the scientific theory of motion by his laws of projectiles, falling bodies and the pendulum. Galileo gave to motion and to time a more precise character than had been allotted by previous thinkers. He undertook to describe more exactly how motion took place, and to give an exact measurable and mathematical character to time. According to Galileo all motions involved certain units of space or

distance covered in certain units of time. This brought space and time into prominence and gave them recognition from a new point of view. Time was no longer the mere measure of motion, but rather something independent of motion, but measured by motion.

The correctness of the Copernican Theory was further proved by John Kepler's laws of planetary movement. Kepler (1571-1630) not only dealt with the geometrical and temporal aspects of planetary motion, but for the first time in the history of human thought attempted to give an account of the mechanics of the movements of the planets.

After a short time René Descartes (1596-1650) came in the field. He looked upon Time as arising from motion, and upon our temporal experience as due to a comparison of the durations of certain regular motions, and as a manner of conceiving of duration in general. Motion was defined by him as "the transporting of one part of matter or of one body from the vicinity of those bodies that are in immediate contact with it, or which we regard at rest, to the vicinity of other bodies." Descartes raised the problem of the relativity of movement. According to Descartes rest and immobility are, and in the nature of things can only be, relative; there are no fixed points in the universe. Descartes' arguments for motion and rest as mutually dependent on systems of reference, and the grasp of the fact that no void or stable point exists by which absolute movement may be calculated, were valuable anticipations of some fundamental points involved in the Theory of Relativity. But in spite of Descartes' interest in motion and its problems, he appears to have been singularly negligent of time as a factor. He drew a distinction between time and duration and suggested that time was only a mode of thinking duration.

However, the latter half of the seventeenth century was marked by great advance in physics and astronomy. At Cambridge, Barrow, who was the predecessor and teacher of Newton, came to treat the problem of time in a new manner. He did not consider time to be identical with motion, nor did he think that time implied motion. He considered that the flow of time was independent of both motion and rest. In order to determine time, we must choose, said Barrow, some mobile which, so far as concerns the periods of its motion, keeps constantly an equal impulse and covers an equal distance. He claimed that time was not motion, but was only measurable by motion. He elaborated at a great length the discussion of the relation of time and motion. But, after all, Barrow formulated a

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definite view of time as a mathematical quantity and was the first to discuss time fully and to carry that discussion, a stage further than the reiteration that time was the measure of motion. He prepared the way not only for his pupil, Sir Issac Newton, but for that very criticism of Newton which came later in the new physics leading up to the doctrine of Relativity.

Thus Newton (1642-1727) came to the field, well equipped with his great teacher's knowledge of the problem, and gave a new tone to orthodox physics which dominated scientific thought for the next two centuries. Newton conceived Real Time as an Absolute, an independent, homogeneous medium, and distinguished this Time, i.e., Absolute Time, which he called duration, from the relative "popular" time, which is a merely sensible and external measure, like hour or month. He defined absolute, true mathematical Time as something which in itself and from its own nature flows equally without relation to anything external. Such absolute Time is a constant, according to Newton. It must be admitted that such a time can be conceived, but whether it is real is quite another matter. It may be a fictitious conception, and this the modern Theory of Relativity asserts. In any case, Newton's Absolute Time presents initial difficulties, if it be true; and this Newton himself admitted when he says that "possibly there is no equable movement that can serve as an exact measure of time". The Absolute Time, therefore, cannot be known or measured, and so far as human perception is concerned must remain unknowable. With regard to the consideration of time-measurement Newton took the earth rotating in absolute space as his fundamental clock. Theoretically even this was not precisely accurate for measuring his "absolute time" unless the whole gravitational force on the earth passed accurately through the centre of mass; but even so, any discrepancy arising from this would not be serious, and could be calculated and allowed for by astronomers after observation over a sufficient number of years. Hence Newton's conception of Time, though severely criticised by the adherents of the Theory of Relativity, has not been discarded by mathematicians as it is sufficient for the purpose of mathematical calculations.

Even among Newton's contemporaries there were many who criticised Newton's Absolute Time and argued that all motion was relative and not absolute. They were of opinion that this true, absolute and mathematical Time was an unnecessary conception, unessential for everyday life and unsuited to the physicist. The Newtonian conception of Absolute Time was severely criticised by

Leibnitz (1646-1716) who though recognised its 'ideal' character in the sense that it was as independent of objects as number was independent of things numbered, yet regarded Absolute Time with its flowing moments as a fiction, and maintained a Relational Theory. Time was for him purely relative and was an order of succession. The criticisms and discussions about Newton's Absolute Time went on for more than a century and the problems of time-measurement, sometimes grasped for solution, gradually developed into a scientific discussion which culminated, later on, in the formulation of the Theory of Relativity.

When the task of time-measurement is undertaken, we have always to resort to something other than time, usually to some movement or set of movements in space. Men employ and have employed in this connection various and varying devices, such as the burning of a candle, the operation of a sand-glass, the shadow of a sun-dial, water-clocks, mechanical clocks, and chronometers, which all measure time by a motion in space. The sun is a general indicator of day and night and of seasonal times. For a field wider than the solar system the velocity of light is employed. No period of time can be measured apart from such methods as these, and our correct estimate of time must, therefore, depend on our correct measurements of space, and these we know only approximately. Astronomers reckon time by the sidereal "clock" based on the time-interval between the passage and repassage of the same star across the meridian. Thus the sidereal day, or the duration of the rotation of the earth as measured by reference to the fixed stars, is the accepted unit for the measurement of time. Even this is not satisfactory, for we have no guarantee that two complete rotations of the earth about its axis have the same duration.

The difficulties inherent in the time-measurement are particularly seen when the use of the word "simultaneity" is investigated. Two sensations are simultaneous when they are perceived instantaneously and not in succession; events are simultaneous as facts of experience just in so far as they are experienced together. But the problem becomes difficult when a physical phenomenon happening at a distance is stated to be simultaneous with an event in a mind remote from the scene of the occurrence. The issue comes to a climax in the treatment Einstein gives to the problem of the determination of simultaneity. Einstein calls two events simultaneous for a given observer when they are perceived or seen at the same time by that observer while he is equidistant from both. This, of course, involves,

and depends upon, the constancy of the velocity of light, that is to say, its speed must be the same in every direction at all times. This is assumed in the definition of simultaneity, but it is an assumption based on electro-magnetic experiments, and the work of Michelson and Morley particularly. But some do not agree with Einstein that such simultaneity is relative and not absolute. The complexities of the problem arise when we endeavour to ascertain whether events which are simultaneous in a given system of reference are also simultaneous in a frame of reference in motion with respect to the first. Arguing from the definition of simultaneity which Einstein has given, he considers whether two flashes of light at points A and B on a railway line are simultaneous. People sitting in a train travelling along the rails with a constant velocity will use the train as a rigid reference-body and they regard all events in reference to the train. If an observer situated half way between A and B sees the two flashes at A and B together, then by definition they are simultaneous events. This is merely an application of the definition. The main difficulty lies in determining whether events which are simultaneous in a given system of reference are also simultaneous in another frame of reference in a state of motion relative to the given system. By examining the above problem Einstein concludes that events which are simultaneous with reference to the railway embankment are not simultaneous with respect to the train, and *vice versa* (relativity of simultaneity) and that every reference-body (co-ordinate system) has its own particular time, unless we are told the reference-body to which the statement of time refers, there is no meaning in a statement of the time of an event. The railway embankment demonstration is a popular exposition of the Special or Restricted Theory of Relativity propounded by Einstein. The whole position there taken up by him follows from, and is a natural development of, the work of Lorentz and his equations of transformation, together with the principle of constancy of the velocity of light.

The motion of time as a fourth dimension was suggested by a friend to D'Alembert as early as 1754 and brought forward by Lange and Lagrange, also by Fechner. As early as 1901, Palagui, a Hungarian, stated his "New Theory of Space and Time," and this was the groundwork of Minkowski's conception of Time as a fourth dimension and that of the Theory of Relativity. Without Minkowski's idea of time as a fourth dimension the general theory of Relativity, as admitted by Einstein, would perhaps have not seen the light at all. It was Minkowski who showed that the Lorentz-Fitzgerald

transformation is equivalent to a twisting of a set of four mutually rectangular axes in a four dimensional space. But this amalgam arrived at by Minkowski is impossible to conceive of except as a system of mathematics. We cannot realise such a world picture, for in it Time has become mathematically symbolized. The space-time of Minkowski and Einstein is but the inevitable outcome of that view of the world begun by Galileo. Time for the mathematical physicist is and must be a measurable time.

In contrast to the Special or Restricted Theory of Relativity the following statement for the general principle of relativity was made by Einstein: All bodies of reference are equivalent for the description of natural phenomena (formulation of the general laws of nature), whatever may be their state of motion. From this principle Einstein proves that in every gravitational field, a clock will go more quickly, according to the position in which the clock is situated (at rest). For this reason, it is not possible to obtain a reasonable definition of time with the aid of clocks which are arranged at rest with respect to the body of reference.

By his theory of relativity Einstein has provoked a revolution of thought in physical science. The achievements consist essentially in this; Einstein has succeeded in separating far more completely than hitherto the share of the observer and the share of external nature in the things we see happen. The perception of an object by an observer depends on his own situation and circumstances; for example, distance will make it appear smaller and dimmer. We make allowance for this almost unconsciously in interpreting what we see. But it now appears that the allowance made for the motion of the observer has hitherto been too crude—a fact overlooked. because in practice all observers share nearly the same motion, that of the earth. Physical space and time are found to be closely bound up with motion of the observer; and only an amorphous combination of the two is left inherent in the external world. When the space and time are relegated to their proper source—the observer—the world of nature which remains appears strangely unfamiliar; but it is in reality simplified and the underlying unity of the principal phenomena is now clearly revealed. Thus Time and Space are entirely relative to the standards of measurement used by observers, and these standards will vary as the relative movements of the systems in which the measurements are made vary. Simultaneity, succession and such similar relations of time and space have, for the new view in contemporary science, no absolute meaning which is eternally and immutably the same everywhere in the universe; on the contrary,

these change from system to system of reference. The work of Mach, Maxwell, Faraday and Poinçare, together with the work of Michelson and Morley, which was followed up by Fitzgerald and Lorentz and finally by Einstein has led to the rejection by the new physics of the Absolute Motion, Time and Space of Newton. But nevertheless for ordinary mathematical calculations Newton's Absolute Time has not been rejected and even with all the discussions on the Mathematical Theory of Relativity by Robb and Eddington, two great exponents of Einstein's Theory, physics has been brought to the border-land of metaphysics by the speculations of Whitehead, Broad, Alexander, Eddington and Russel on the problem of Time.

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SOME PROPERTIES OF DEMLO NUMBERS AND THEIR APPLICATIONS TO RECURRING DECIMALS

BY

D. R. KAPREKAR, *Khare's Wada, Devlali.*

1. The object of this paper is two-fold. Firstly I propose to use the theory of Demlo numbers to give a solution of Question No. 1758 in Vol. VI, pp. 85-86 of *The Mathematics Student.* Next, I discuss a method by which the recurring periods in the decimal representation of the reciprocals of prime numbers can be obtained by multiplication instead of the ordinary process of division. Solutions by others for the Question No. 1758 are found in the above reference.

2. As pointed out elsewhere*, a Demlo number consists of three parts, the sum of the first and the last parts being a number with a single repeated digit while the middle part is either empty or is also a number with the same repeated digit. For instance, 231768 and 23199768 are both Demlo numbers. I shall first prove two general results which are used later to discuss the problems mentioned in § 1.

2·1. THEOREM 1. *If N is a number with k digits and P a number consisting of l $(l \geqslant k)$ repeated nines, then $N \times P$ is a Demlo number in which the first part is $N-1$ and the middle part consists of $(l-k)$ repeated nines.*

---

* *The Mathematics Student*, 6 (1938), pp. 68-70.

Proof: Let

$$N = a_0\, 10^{k-1} + a_1\, 10^{k-2} + \cdot \cdot + a_{k-1}$$

$$P = 10^l - 1 = 999 \cdot \cdot (l \text{ times}).$$

Then

$$\begin{aligned} N \times P &= a_0\, 10^{k+l-1} + \cdot \cdot + a_{k-1}\, 10^l - a_0\, 10^{k-1} - \cdot \cdot - a_{k-1} \\ &= a_0\, 10^{k+l-1} + \cdot \cdot + (a_{k-1} - 1)\, 10^l + 9 \cdot 10^{l-1} + \cdot \cdot + 9 \cdot 10^k \\ &\quad + (10 - a_0 - 1)\, 10^{k-1} + \cdot \cdot + (10 - a_{k-2} - 1)\, 10 + (10 - a_{k-1}). \end{aligned}$$

The last relation proves the theorem.

2·2. THEOREM 2. *Let N be a number having a zero in its last digit. Then in the product*

$$Z = (N-1)\,(1 + N + N^2 + \cdot + N^r),$$

the last $(r+1)$ *digits will be nines.*

Proof: We have

$$Z = (N-1) \cdot \frac{N^{r+1} - 1}{N - 1} = N^{r+1} - 1$$

and the result follows at once.

2·3. As illustrations of theorem 1 we have

$$5432 \times 99999 = 543194568$$

$$673 \times 999 = 672327.$$

As illustrations of theorem 2 we have

$$139\ (1 + 140 + 140^2 + 140^3 + 140^4)$$

$$= 50722499999$$

3. In Question 1758 mentioned in § 1 we have to prove that the last nine digits in

$$(546194538)\{1 + 24580 + (24580)^2 + \cdot \cdot + (24580)^{10}\}$$

are 999977778. To prove this, we note by theorem 1 and inspection, that

$$\begin{aligned} 546194538 &= 5462 \times 99999 \\ &= 2 \times 2731 \times 11111 \times 9 \\ &= 24579 \times 22222 \end{aligned}$$

Hence taking $N = 24580$ in theorem 2, we conclude that the last 11 digits in

$$24579\ \{1 + 24580 + (24580)^2 + \cdot \cdot + (24580)^{10}\}$$

are nines. Hence the product required is 22222 $\{P+Q\}$ where P contains 11 zeros at the end while Q is the number $10^{11}-1=99\cdots$ (11 times). By theorem 1, the product $22222\times Q$ is equal to

$$2222199999977778$$

and since 22222 P contains 11 zeros at the end it follows that the last nine digits in the product are those stated in the problem.

3·1. We can use a similar method for the following problems as well:

(1) Prove that the last 8 digits in

$$238053\ \{1+220+(220)^2-\cdots+(220)^8\}\quad \text{are}\quad 99998913.$$

(2) Prove that the last ten digits in

$$1375961\ \{1+140+(140)^2+\cdots+(140)^{10}\}\ \text{are}\ 9999990101.$$

4. We next proceed to discuss the second problem mentioned in § 1. The clue to the method followed in this connection is found in theorem 2. We shall first discuss a special case and then point out the general method. We shall first explain how to obtain the recurring part of the fraction $\frac{1}{7}$ by multiplication instead of division. We find the least number a such that $7a+1$ is divisible by 10. Here $a=7$. Taking $N=7a+1=50$ in Theorem 2 we find that the last $(r+1)$ digits in

$$50^{r+1}-1=49\ \{1+50+50^2+\cdots+50^r\}$$

must be nines. Since 10^6-1 is divisible by 7 we see that the recurring portion in the fraction $\frac{1}{7}$ will be the last 6 digits in

$$\frac{50^{r+1}-1}{7}=7\ \{1+50+50^2+\cdots+50^r\}$$

for $r\geqslant 6$. This is so because if we write the above equation as

$$\frac{50^{r+1}-1}{7}=a_0\,a_1\,a_2\cdots a_p$$

we get by dividing both sides by 50^{r+1}

$$\frac{1}{7}=\frac{a_0\,a_1\cdots a_p}{50^{r+1}}+\frac{1}{7\cdot 50^{r+1}}$$

and so the last 6 digits in $a_0\,a_1\cdots a_p$ must represent the recurring

part in the fraction 1/7. Taking $r=6$ and working out the multiplication $7\,(1+50+50^2+\cdot+50^6)$ in detail we get

$$\begin{array}{r}
7\\
3\,5\,0\\
1\,7\,5\,0\,0\\
8\,7\,5\,0\,0\,0\\
4\,3\,7\,5\,0\,0\,0\,0\\
2\,1\,8\,7\,5\,0\,0\,0\,0\,0\\
1\,0\,9\,2\,7\,5\,0\,0\,0\,0\,0\,0\\
\hline
\cdots\cdot\,1\,4\,2\,8\,5\,7
\end{array}$$

Hence the recurring portion is 142587. Of course if we do not give any definite value for r but continue the multiplication by powers of 50 indefinitely we shall find that the portion 142587 is repeated again and again.

4·1. We now proceed to indicate the method in the general case. Let p be any odd prime number other than 5. Let a be the least integer such that $N=ap+1$ is divisible by 10. Then the number

$$\begin{aligned} N^{r+1}-1 &= (N-1)(1+N+N^2+\cdots\cdot+N^r)\\ &= ap\,(1+N+N^2+\cdots\cdots+N^r) \end{aligned}$$

is divisible by p and has the last $r+1$ digits nines in virtue of theorem 2. Since $10^{p-1}-1$ is divisible by p it follows that a number with $(p-1)$ nines will be divisible by p. Hence, as before, if $r \geqslant p-1$, the last $p-1$ digits in

$$\frac{N^{r+1}-1}{p} = a(1+N+N^2+\cdots\cdot+N^r)$$

will represent a recurring period of the fraction $1/p$. These $p-1$ digits will either be the *actual* recurring period of $1/p$ or will contain the recurring portion a *whole number of times*. In practice, it is better to allow r to be indefinitely large and then, as the multiplication is carried on, the recurring portion will appear again and again.

4·2. We can also give the value of a in the various possible cases. We can take $a=9$, 3, 7, or 1 according as the prime ends in 1, 3, 7 or 9. We shall give another illustration with $p=19$. Here $a=1$ and we have to find the last 18 digits in

$$1\cdot(1+20+20^2+20^3+\ldots) \qquad \ldots \quad (1)$$

Noting that at each stage in the multiplication by 20 we have to add a zero and multiply by 2, the process of writing down each product

of the form $1 \cdot 20^k$ and adding up can be carried out shortly as follows :—

$$\overline{\cdots \cdot 21052631578947368421}$$

$$\overline{\cdots \cdot 0101001111010110000}$$

The recurring period is 052631578947368421. In the above process, we multiply 1 the first term in (1) by 2, put the unit figure and carry the rest (here, zero). Next we multiply 2 by 2 and proceed as in ordinary multiplication. Then $2 \times 1 = 2$, $2 \times 2 = 4$, $2 \times 4 = 8$, $2 \times 8 = 16$ (here we put 6 and carry 1), $2 \times 6 = 12 + 1 = 13$ (here we put 3 and carry 1) and so on. The carrying figure at each stage of multiplication is indicated in the lower row while the figures appearing at each stage of multiplication is shown in the top row. This process can also be applied to the product for $\frac{1}{7}$, namely

$$7(1 + 50 + 50^2 + 50^3 + \dots \dots).$$

Here we first put 7 and multiply by 5 successively. Thus

$$\overline{\cdots \cdot 7142857}$$

$$\overline{\cdots \cdot 214230}$$

This method can be applied to find successive digits of any product of the form

$$a(1 + N + N^2 + \dots \dots)$$

where N is a number ending with zero. The number a may be called the first figure and the number N from which the last zero is cancelled may be called the constant multiplier and the process itself may be called the carrying forward process. The method can be applied to any odd prime other than 5.

THE CONIC IN HOMOGENEOUS CO-ORDINATES

BY

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1. In this paper I discuss the geometry of the general equation of the second degree in a system of homogeneous co-ordinates in which the circular points are given by an assigned equation of the first degree and another of the second degree.

2. Let the equation of the line at infinity be

$$ax + by + cz = 0$$

(a, b, c not being the sides of the triangle of reference), and let

$$\theta(x,y,z) \equiv \sum u_0 x^2 + 2\sum f_0 yz = 0$$

be the equation of a fixed circle in the plane.

Let the equation of a given conic be

$$\varphi(x, y, z) \equiv \sum ux^2 + 2 \sum fyz = 0.$$

Let $X = \frac{1}{2}\frac{\partial\varphi}{\partial x}$, $Y = \frac{1}{2}\frac{\partial\varphi}{\partial y}$, $Z = \frac{1}{2}\frac{\partial\varphi}{\partial z}$,

and $\xi = cY - bZ$, $\eta = aZ - cX$, $\zeta = bX - aY$,

so that $$a\xi + b\eta + c\zeta \equiv 0 \quad \ldots \quad (2\cdot1)$$

Also put

$$p = vc^2 + wb^2 - 2bcf,\ q = wa^2 + uc^2 - 2acg,\ r = ub^2 + va^2 - 2abh, \quad \ldots \quad (2\cdot2)$$

and

$$-a^2p + b^2q + c^2r = 2bc\lambda, \quad a^2p - b^2q + c^2r = 2ca\mu$$
$$a^2p + b^2q - c^2r = 2ab\nu \quad \ldots \quad (2\cdot3)$$

whence we easily get

$$ap = c\mu + b\nu, \qquad bq = a\nu + c\lambda, \qquad cr = b\lambda + a\mu, \quad \ldots \quad (2\cdot4)$$

which give

$$\begin{vmatrix} -p & \nu & \mu \\ \nu & -q & \lambda \\ \mu & \lambda & -r \end{vmatrix} = 0$$

Let Δ stand for the determinant

$$\begin{vmatrix} u & h & g \\ h & v & f \\ g & f & w \end{vmatrix}$$

The following identities can be easily proved.

$$\begin{vmatrix} u & h & g & a \\ h & v & f & b \\ g & f & w & c \\ a & b & c & o \end{vmatrix} = \frac{1}{4a^2b^2c^2}\begin{vmatrix} o & cr & bq & a \\ cr & o & ap & b \\ bq & ap & o & c \\ a & b & c & o \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} \lambda & o & o & a \\ o & \mu & o & b \\ o & o & \nu & c \\ 1 & 1 & 1 & o \end{vmatrix}$$

i. e. $\sum Ua^2 + 2\sum Fbc \equiv \frac{-1}{4a^2b^2c^2}(\sum a^4p^2 - 2\sum b^2c^2qr) = \frac{1}{abc}\sum a\,\mu\,\nu$ ... (2·6)

where U, V etc., are the co-factors of u, v, etc. in Δ.

Thus the conic φ will be an ellipse, parabola, or hyperbola according as

$$\sum Ua^2 + 2\sum bcF > = \text{or} < 0 \text{ or } \sum a^4p^2 - 2\sum b^2c^2qr < = \text{or} > 0$$

$$\text{or } \sum a\,\mu\,\nu > = \text{or} < 0 \qquad \text{... (2·7)}$$

In particular, if the conic φ is a parabola $\sum a/\lambda = 0$, and the point at infinity on it is $(1/\lambda, 1/\mu, 1/\nu)$.

3. Lemma.

The joint equation of the points at infinity on the conic $\varphi(x,y,z)=0$ is

$$\sum pl^2 - 2\sum\lambda mn = 0 \qquad \text{... (3·1)}$$

In particular, the joint equation of the circular points I and J at infinity is

$$\sum p_o\, l^2 - 2\sum\lambda_o\, mn = 0 \qquad \text{... (3·2)}$$

where p_o, λ_o, etc. are the functions of the coefficients of $\theta\,(x, y, z)=0$ similar to p, λ, etc. (2·2, 2·3)

Proof: The points at infinity P (x_1, y_1, z_1), Q (x_2, y_2, z_2) on φ satisfy the equations

$$\varphi\ (x,\ y,\ z) = 0,\ ax + by = cz = 0$$

$$\therefore \qquad qx^2 + 2\nu xy + py^2 = 0$$

and two similar equations obtained by eliminating x and y. Hence

$$\frac{x_1x_2}{p} = \frac{y_1y_2}{q} = \frac{z_1z_2}{r} = \frac{y_1z_2 + y_2z_1}{-2\lambda} = \frac{z_1x_2 + z_2x_1}{-2\mu} = \frac{x_1y_2 + x_2y_1}{-2\nu} \qquad \text{... (3·3)}$$

whence the lemma follows. The second part follows from the first.

4. Circles connected with the triangle of reference.

Circumcircle.

Writing the equation $\theta\ (x,\ y,\ z)=0$ in the form

$$\sum ax \sum \frac{u_o}{a}x = \frac{1}{abc}\sum ap_o yz,$$

we see that the line at infinity meets the circle $\theta\ (x,\ y,\ z)=0$ in points which lie on the circum-conic $\sum ap_o yz=0$. Thus the equation of the circumcircle of the triangle of reference is

$$\sum ap_o yz=0 \qquad \dots \quad (4{\cdot}1)$$

Polar circle.

The *polar conic* $\sum t_1\ x^2=0$, will be a circle if it passes through the points I $(x_1,\ y_1,\ z_1)$. J $(x_2,\ y_2,\ z_2)$, i.e. if $\sum t_1\ x_1{}^2=0$, $\sum t_1\ x_2{}^2=0$, whence $t_1 : t_2 : t_3 = a\lambda_o : b\mu_o : c\nu_o$.

Hence the equation of the polar circle is

$$\sum a\lambda_o x^2=0 \qquad \dots \quad (4{\cdot}2)$$

Conditions for a circle.

Writing the equation $\varphi\ (x,\ y,\ z)=0$ in the form

$$\sum ax \sum \frac{u}{a}x = \frac{1}{abc}\sum apyz,$$

we see that the conic φ will represent a circle, if the right-hand side equated to zero represents the circumcircle. Hence

$$\frac{p}{p_o}=\frac{q}{q_o}=\frac{r}{r_o} \qquad \dots \quad (4{\cdot}3)$$

and each one of these ratios is equal to

$$\frac{\lambda}{\lambda_o}=\frac{\mu}{\mu_o}=\frac{\nu}{\nu_o}$$

The result also follows from (3·1) and (3·2)

5. The joint equation of the polars w. r. t. ϕ of the points at infinity on the conic $\psi\ (x,\ y,\ z) \equiv \sum u'x^2+2\sum f'yz=0$ can be written in either of the following forms :—

$$\sum p'X^2 - 2\sum \lambda' YZ=0 \qquad \dots \quad (5{\cdot}1)$$

$$\psi\ (\xi,\ \eta,\ \zeta)=0 \qquad \dots \quad (5{\cdot}2)$$

$$\sum a\lambda'\xi^2 = 0 \qquad \dots \quad (5{\cdot}3)$$

$$\sum ap'\eta\zeta = 0 \qquad \dots \quad (5{\cdot}4)$$

where p', q', r', λ', μ', ν' are functions of the coefficients in $\psi\ (x,\ y,\ z)=0$ similar to p, q etc. (2·2, 2·3)

To prove the first relation, suppose R (x', y', z'), S (x'', y'', z'') are the points at infinity on ψ. The joint equation of their polar w. r. t. φ is

$$\sum x'X \ \sum x''X = 0,$$

which in virtue of (3·3) reduces to

$$\sum p'X^2 - 2 \ \sum \lambda' \ YZ = 0$$

To prove the second result, suppose that (x', y', z') is a point on either polar. The polar of (x', y', z') is

$$xX' + yY' + zZ' = 0$$

and the point at infinity on it is (ξ', η', ζ'). This lies on $\psi = 0$, therefore $\psi(\xi', \eta', \zeta') = 0$. Hence the equation of the polars is

$$\psi(\xi, \eta, \zeta) = 0.$$

The last two results are easy deductions from $\psi(\xi, \eta, \zeta) = 0$ in virtue of $\sum a\xi = 0$

It is obvious that these lines are diameters of φ, since ξ, η, ζ vanish at the centre of φ.

The equations $\sum a\lambda'\xi^2 = 0$, $\sum ap'\eta\zeta = 0$ are polars w. r. t. φ of the points at infinity on the conics $\sum a\lambda' x^2 = 0$, $\sum ap'yz = 0$ respectively. These points also lie on ψ; it follows therefore that the three conics

$$\psi = 0, \ \sum a\lambda' x^2 = 0, \ \sum ap'yz = 0$$

have a pair of common points at infinity.

6. Asymptotes.

The polars w. r. t. φ of the points at infinity on φ are its asymptotes. Hence the joint equations of the asymptotes of $\varphi(x, y, z) = 0$ can be written in either of the forms

$$\sum pX^2 - 2\sum \lambda YZ = 0 \quad \ldots \quad (6\cdot1)$$

$$\varphi(\xi, \eta, \zeta) = 0 \quad \ldots \quad (6\cdot2)$$

$$\sum a\lambda\xi^2 \equiv - \sum ap\eta\zeta = 0 \quad (6\cdot3)$$

The three conics $\varphi(x, y, z) = 0$, $\sum a\lambda x^2 = 0$, $\sum apyz = 0$ have a pair of common points at infinity.

7. The points at infinity on $\psi(x, y, z) = 0$ are conjugate w. r. t. $\varphi(x, y, z) = 0$, if

$$\sum up' - 2\sum f'\lambda' \equiv \sum u'p - 2\sum f'\lambda = 0 \quad \ldots \quad (7\cdot1)$$

$$\text{or} \quad \sum a\lambda p' = 0 \quad \ldots \quad (7\cdot2)$$

$$\text{or} \quad \sum a\lambda' p = 0 \quad \ldots \quad (7\cdot3)$$

If $R(x', y', z')$, $S(x'', y'', z'')$ are the points at infinity on $\psi=0$ and P, Q on φ, then if R and S are conjugate w.r.t. φ, $(PQ, RS)=-1$, and therefore R and S are conjugate w.r.t. every conic through P and Q, and P and Q are conjugate w.r.t. every conic through R and S. In particular, R and S will be conjugate w. r. t. the conics $\varphi=0$, $\sum a\lambda x^2=0$, $\sum apyz=0$, hence

$$\sum ux'x''+\sum f(y'z''+y''z')=0,\ \ \sum a\lambda x'x''=0,\ \ \sum ap(y'z''+y''z')=0$$

from which follow the relations (7·1), (7·2), (7·3). The second part of (7·1) follows from the fact that P and Q are conjugate w.r. to ψ.

8. Rectangular Hyperbola.

The conic φ will be a rectangular hyperbola, if its points at infinity P, Q are harmonically separated by I and J. Thus P and Q (I and J) will be conjugate w. r. t. every conic through I and J (P and Q). Thus replacing $\psi(x, y, z)=0$ in (7) by $\theta(x, y, z)=0$, we get the required conditions

$$\sum up_o-2\sum f\lambda_o \equiv \sum u_op-2\sum f_o\lambda=0 \qquad \ldots \quad (8\cdot1)$$

$$\sum a\lambda p_o=0 \qquad \ldots \quad (8\cdot2)$$

$$\sum a\lambda_op=0 \qquad \ldots \quad (8\cdot3)$$

9. Conjugate diameters.

The equation of a pair of diameters of φ can be reduced to either of the forms

$$t_1\xi^2+t_2\eta^2+t_3\zeta^2=0 \qquad \ldots \quad (9\cdot1)$$

$$T_1\eta\zeta+T_2\zeta\xi+T_3\xi\eta=0 \qquad \ldots \quad (9\cdot2)$$

and these lines are the polars w.r.t. φ of the points at infinity on the conics $\sum t_1x^2=0$, $\sum T_1yz=0$ respectively. The lines (9·1) will be conjugate diameters of φ, if the points at infinity P, Q on φ are conjugate w. r. t. $\sum t_1x^2=0$,

$$\therefore \quad t_1p+t_2q+t_3r=0 \qquad \ldots \quad (9\cdot3)$$

Similarly, the lines (9·2) will be conjugate diameters of φ if

$$T_1\lambda+T_2\mu+T_3\nu=0 \qquad \ldots \quad (9\cdot4)$$

Thus the equations

$$\sum \frac{t_2-t_3}{p}\xi^2=0 \qquad \sum \frac{T_2-T_3}{\lambda}\eta\zeta=0 \qquad \ldots \quad (9\cdot5)$$

represent the conjugate diameters of φ.

10. Axes of the Conic φ.

The Axes of a conic are the unique pair of perpendicular conjugate diameters. Thus the conjugate diameters $\sum \frac{t_2 - t_3}{p} \xi^2 = 0$ which are the polars w. r. t. φ of the points at infinity on the conic $\sum \frac{t_2 - t_3}{p} x^2 = 0$ will be perpendicular, if their points at infinity which coincide with the points at infinity on $\sum \frac{t_2 - t_3}{p} x^2 = 0$ are harmonically separated by I and J, i.e., if I and J are conjugate w. r. t. $\sum \frac{t_2 - t_3}{p} x^2 = 0$ and this requires

$$\sum p_o \frac{t_2 - t_3}{p} = 0.$$

Thus eliminating $\frac{t_2 - t_3}{p}, \frac{t_3 - t_1}{q}, \frac{t_1 - t_2}{r}$ from equations $\sum \frac{t_2 - t_3}{p} \xi^2 = 0$, $\sum \frac{t_2 - t_3}{p} p_o = 0$ and $\sum \frac{t_2 - t_3}{p} p = 0$, we get the equation of the axes in the form

$$\begin{vmatrix} \xi^2 & \eta^2 & \zeta^2 \\ p & q & r \\ p_o & q_o & r_o \end{vmatrix} = 0 \qquad \dots \quad (10{\cdot}1)$$

The equation of the axes can also be written in the form

$$\begin{vmatrix} \eta\zeta & \zeta\xi & \xi\eta \\ \lambda & \mu & \nu \\ \lambda_o & \mu_o & \nu_o \end{vmatrix} = 0 \qquad \dots \quad (10{\cdot}2)$$

11. Axes of a parabola.

The polar of the point at infinity (x', y', z'), w.r.t. the parabola φ will be its axis if (x', y', z') and the point at infinity $(1/\lambda, 1/\mu, 1/\nu)$ on the parabola which coincides with the point at infinity on its axis are conjugate w.r.t. a circle, say $\sum a\lambda_o x^2 = 0$. Thus the line

$$x'X + y'Y + z'Z = 0.$$

will be the axis of the parabola φ, if

$$ax' + by' + cz' = 0$$

$$a\lambda_o \frac{x'}{\lambda} + b\mu_o \frac{y'}{\mu} + c\,\nu_o \frac{z'}{\nu} = 0$$

Thus the equation of the axis of the parabola φ is

$$\begin{vmatrix} X & Y & Z \\ a & b & c \\ \frac{a\lambda_o}{\lambda} & \frac{b\mu_o}{\mu} & \frac{c\nu_o}{\nu} \end{vmatrix} = 0$$

Or $$\sum a \frac{\lambda_o}{\lambda} \xi = 0 \qquad \ldots \quad (11\cdot1)$$

12. Director Circle.

Let I and J be the points at infinity common to all circles in the plane. The pencil of conjugate rays w. r. t. φ through I and J intersect on a conic through I and J and is, therefore, a circle. It is called the director circle. It can easily be proved that the tangents to φ from any point of its director circle are perpendicular.

Let the co-ordinates of I and J be respectively $(x_1, y_1\ z_1)$, (x_2, y_2, z_2). Suppose (x, y, z) are the co-ordinates of the point of intersection of two conjugate rays from I and J. The co-ordinates of these rays are respectively

$$(yz_1 - y_1z,\ zx_1 - z_1x,\ xy_1 - x_1y),\ (yz_2 - y_2z,\ zx_2 - z_2x,\ xy_2 - x_2y).$$

These lines are conjugate w. r. t. φ if

$$\sum U(yz_1 - y_1z)(yz_2 - y_2z) + \sum F[(zx_1 - z_1x)(xy_2 - x_2y) + (zx_2 - z_2x)(xy_1 - x_1y)] = 0$$

or $$\sum U(r_oy^2 + q_oz^2 + 2\lambda_oyz) - 2\sum F(p_oyz + \nu_ozx + \mu_oxy - \lambda_ox^2) = 0$$

or $$\sum (Vr_o + Wq_o + 2F\lambda_o)x^2 - 2\sum (p_oF + \nu_oG + \mu_oH - \lambda_oU)yz = 0 \qquad \ldots \quad (12\cdot1)$$

The equation can be written in the standard form

$$\sum ax \sum \frac{Vr_o + Wq_o + 2F\lambda_o}{a} x = \frac{k}{abc} \sum ap_oyz. \qquad \ldots \quad (12\cdot2)$$

where $k \equiv \sum a^2U + 2\sum bc\ F$.

If the conic φ is a parabola $k = 0$, and the equation

$$\sum \frac{Vr_o + Wq_o + 2F\lambda_o}{a} x = 0 \qquad \ldots \quad (12\cdot3)$$

will give the directrix of the parabola φ.

Second Form.

Let the lines be the polars of the points R (x', y', z'), S (x'', y'', z'') w. r. t. φ, so that R and S are conjugate w. r. t. φ. Suppose that

the polar of R passes through I (x_1, y_1, z_1) and the polar of S passes through J (x_2, y_2, z_2). If (x, y, z) be a point on the director circle, then the following equations are simultaneously satisfied.

$$x'X + y'Y + z'Z = 0 \qquad x''X + y''Y + z''Z + 0$$

$$x'X_1 + y'Y_1 + z'Z_1 = 0 \qquad x''X_2 + y''Y_2 + z''Z_2 = 0$$

$$x''X' + y''Y' + z''Z' = 0$$

From the last three relations we get

$$\begin{vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \\ X' & Y' & Z' \end{vmatrix} = 0 \qquad \ldots \quad (12\cdot4)$$

$$\text{or } 0 = \begin{vmatrix} \sum x_1X & \varphi - xX & Z \\ \sum x_1X_2 & \sum x_2X - xX_2 & Z_2 \\ 0 & -xX' & Z' \end{vmatrix} = \begin{vmatrix} \sum x_1X & \varphi & Z \\ \sum x_1X_2 & \sum x_2X & Z_2 \\ 0 & 0 & Z' \end{vmatrix} - x \begin{vmatrix} \sum x_1X & X & Z \\ \sum x_1X_2 & X_2 & Z_2 \\ 0 & X' & Z' \end{vmatrix}$$

or $$\sum x_1X_2 \cdot \varphi(x, y, z) = \sum x_1X \sum x_2X$$

since the last determinant vanishes, (12·4) and this equation reduces to

$$(\sum up_o - 2\sum f\lambda_o)\, \varphi(x, y, z) - \sum p_oX^2 - 2\sum \lambda\, YZ \qquad \ldots \quad (12\cdot5)$$

The same result can be obtained from the consideration that the pair of tangents from a point on the director circle are perpendicular. Let $(x'y'z')$ be a point in the plane of the conic. The equations of the pair of tangents from (x', y', z') to φ is

$$\varphi(x, y, z)\, \varphi(x', y', z') = (xX' + yY' + zZ')^2$$

i.e. $$\sum (u\varphi' - X'^2)\, x^2 + 2\sum (f\varphi' - Y'Z')yz = 0$$

If $(x_1, y_1, z_1,)$ (x_2, y_2, z_2) be the points at infinity on these tangents

$$\frac{x_1\ x_2}{p\varphi' - \xi'^2} = \frac{y_1\ y_2}{q\varphi' - \eta'^2} = \frac{z_1\ z_2}{r\varphi' - \zeta'^2}$$

The points (x_1, y_1, z_1), (x_2, y_2, z_2) are conjugate w. r. t. the polar circle $\sum a\lambda_o\, x^2 = 0$ if

$$\sum a\lambda_o\, (p\varphi' - \xi'^2) = 0.$$

Hence $(x'\ y'\ z')$ lies on the locus

$$\varphi\,(x,\ y,\ z)\ \sum ap\lambda_o = \sum a\lambda_o\xi^2 \qquad \ldots \quad (12{\cdot}6)$$

which can also be written in the form

$$\varphi\,(x,\ y,\ z)\ \sum ap\lambda_o = -\sum ap_o\eta\zeta \qquad \ldots \quad (12{\cdot}7)$$

It may be remarked that the right-hand sides equated to zero of equations (12·5), (12·6), (12·7) represent the polars w. r. t. φ of the circular points at infinity, and the coefficients of $\varphi\ (x,\ y,\ z)$ on the left equated to zero are two different forms of condition for a rectangular hyperbola (8·1, 8·3).

13. Foci.

If the point $(x',\ y',\ z')$, of the previous section be a focus, the points $(x_1,\ y_1,\ z_1)$, $(x_2,\ y_2,\ z_2)$ at infinity on the tangents from $(x',\ y',\ z')$ coincide with I and J. Thus

$$\frac{p\varphi' - \xi'^2}{p_o} = \frac{q\varphi' - \eta'^2}{q_o} = \frac{r\varphi' - \eta'^2}{r_o}.$$

Thus the foci lie on the conics

$$\frac{p\varphi - \xi^2}{p_o} = \frac{q\varphi - \eta^2}{q_o} = \frac{r\varphi - \zeta^2}{r_o} \qquad \ldots \quad (13{\cdot}1)$$

Directrices. The directrices of a conic φ are the degenerate conics of the system generated by the given conic and the polars of I and J w. r. t. φ, viz.

$$k\varphi = \sum p_o\,X^2 - 2\ \sum \lambda_o\,YZ \qquad \ldots \quad (14{\cdot}1)$$

where k is the parameter of the system. To determine k so that (14·1) may represent a pair of lines, we transform the above equation by the identity

$$\triangle\ \varphi = \sum UX^2 + 2\sum FYZ.$$

The equation (14·1) becomes

$$\sum (kU - p_o\triangle)\ X^2 + 2\sum (kF + \lambda_o\triangle)\ YZ = 0$$

This will be a product of two linear factors if

$$\begin{vmatrix} kU - p_o\triangle & kH + \triangle\nu_o & kG + \triangle\mu_o \\ kH + \triangle\nu_o & kV - \triangle q_o & kF + \triangle\lambda_o \\ kG + \triangle\mu_o & kF + \triangle\lambda_o & kW - \triangle r_o \end{vmatrix} = 0$$

i.e. $\quad k^2 - \delta_1 k + \delta_2 = 0 \qquad \ldots \quad (14{\cdot}2)$

where $\quad \delta_1 \equiv \sum p_oU - 2\sum F\lambda_o$ and $\delta_2 \equiv (\sum a^2U_o + 2\sum bcF_o)(\sum a^2U + 2\sum bcF)$.

A NOTE ON THE RATIO OF THE IN- AND CIRCUM-RADII OF A TETRAHEDRON

BY

P. K. KASHIKAR, *Wilson College, Bombay.*

1. In Question No. 1763, in *The Mathematics Student* (December 1937), it is required to prove that the inradius r of a tetrahedron is less than half the circumradius R. While solving this problem, I found that the maximum value of the ratio r/R for a tetrahedron is 1/3, and for a simplex with $(n+1)$ vertices in n-dimensional space this value is $1/n$. The present note contains a simple proof of these results.

2. To prove the theorem for a tetrahedron we require the following:

Lemma: If the base ABC of a tetrahedron is fixed, and the fourth vertex D moves at a constant distance h from the plane ABC, then the maximum value of the inradius is $\dfrac{hr_0}{r_0+\sqrt{(h^2+r_0^2)}}$, where r_0 is the inradius of $\triangle$ ABC, and this value is attained when the foot of the altitude through D coincides with the in-centre of $\triangle$ ABC.

Proof: Let $\triangle_A, \triangle_B, \triangle_C, \triangle_D$ denote the areas of the faces of the tetrahedron; a, b, c the sides of ABC; and α, β, γ the trilinear co-ordinates of the foot of the altitude through D.

Then $r = 3V \div (\triangle_A + \triangle_B + \triangle_C + \triangle_D)$; and since V and $\triangle_D$ are constant, r will be a maximum when $\triangle_A + \triangle_B + \triangle_C$ is a minimum.

Now $2(\triangle_A + \triangle_B + \triangle_C) = a\sqrt{h^2+\alpha^2} + b\sqrt{h^2+\beta^2} + c\sqrt{h^2+\gamma^2}.$

and $2\triangle_D = a\alpha + b\beta + c\gamma$

Using the method of undetermined multipliers, we find that the expression $\triangle_A + \triangle_B + \triangle_C$ has a stationary value, when

$$\frac{\alpha}{\sqrt{h^2+\alpha^2}} = \frac{\beta}{\sqrt{h^2+\beta^2}} = \frac{\gamma}{\sqrt{h^2+\gamma^2}}$$

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i.e. when $\alpha=\beta=\gamma=r_o$ when the foot of the altitude through D coincides with the incentre of ABC

Writing r_o+x, r_o+y, r_o+z for α, β, γ respectively, and considering the terms of the second degree in x, y, z in the expansion of $\sum a\sqrt{h^2+\alpha^2}$, we can show that for this position of the foot of the altitude, the expression $\Delta_A+\Delta_B+\Delta_C$ is a minimum and therefore r is a maximum. Also in this case

$$r=\frac{3V}{\Delta_A+\Delta_B+\Delta_C+\Delta_D}=\frac{\frac{1}{2}\,h\,r_o\,(a+b+c)}{\frac{1}{2}\,(a+b+c)\,\sqrt{h^2+r_o^2}+\frac{1}{2}r_o\,(a+b+c)}$$

$$=\frac{hr_o}{\sqrt{h^2+r_o^2}+r_o}$$

In the same way, for a simplex in n-dimensional space, it can be shown that, if n out of the $(n+1)$ vertices are fixed, and the $(n+1)^{\text{th}}$ vertex moves at a constant distance h from the $(n-1)$-flat passing through the n fixed vertices, then the maximum value of the inradius is $\dfrac{hr_o}{r_o+\sqrt{h^2+r_o^2}}$ where r_o is the inradius of the fixed base, and this value is attained when the foot of the altitude through the $(n+1)^{\text{th}}$ vertex, coincides with the incentre of the base.

3. Now take any tetrahedron ABCD, inscribed in a given sphere of radius R. Let r_o and R_o be the in- and circum-radii of $\triangle$ ABC, and let I and O be the in- and circum-centres (of $\triangle$ ABC).

Take any equilateral triangle $A_oB_oC_o$ having the same circumcircle as $\triangle$ ABC, and therefore having its in- and circum-centres coincident with O.

Draw a plane through D parallel to the plane ABC, and let the perpendiculars at I and O to the plane ABC, meet the parallel plane through D in the points D_1 and D_2, and let OD_3 produced meet the sphere ABCD in D_o.

Then, the inradius of the equilateral triangle $A_oB_oC_o$ is greater than r_o, and since the expression

$$\frac{hr_o}{r_o+\sqrt{h^2+r_o^2}}=1\div\left\{\frac{1}{h}+\sqrt{\left(\frac{1}{r_o^2}+\frac{1}{h^2}\right)}\right\}$$ increases with h and r_o

we see that $\left(\begin{array}{c}\text{inradius}\\ \text{of DABC}\end{array}\right)<\left(\begin{array}{c}\text{inradius}\\ \text{of } D_1ABC\end{array}\right)<\left(\begin{array}{c}\text{inradius}\\ \text{of } D_2A_oB_oC_o\end{array}\right)<\left(\begin{array}{c}\text{inradius}\\ \text{of } D_oA_oB_oC_o\end{array}\right)$

The ratio r/R is greater for the tetrahedron $D_oA_oB_oC_o$ than for the original tetrahedron DABC, since r/R has an upper bound which is actually attained. This means that so long as any face of a tetrahedron is not an equilateral triangle, and the foot of the corresponding altitude does not coincide with its incentre, it is possible to increase the ratio r/R.

It follows that this ratio will be a maximum when the tetrahedron is regular, and then its value is $\frac{1}{3}$.

4. The corresponding result for the general simplex $A_1A_2 \cdots A_{n+1}$ can now be proved by induction.

Let $B_1B_2 \cdots B_n$ be a regular $(n-1)$-dimensional simplex, having the same circumscribed hypersphere as $A_1A_2 \cdots A_n$. Let O be the centre of this regular simplex, and let B_{n+1} be the point, in which the normal at O to the $(n-1)$-flat $A_1A_2 \cdots A_n$ meets the circumscribed hypersphere of the simplex $A_1 A_2 \cdots A_n A_{n+1}$. Assuming the theorem to be true for an $(n-1)$-dimensional simplex, it can be shown exactly as in the case of the tetrahedron, that the inradius of $B_1B_2 \cdots B_n B_{n+1}$ is greater than the inradius of $A_1 A_2 \cdots A_n A_{n+1}$; and then it follows, exactly as before, that the ratio r/R is a maximum when the simplex is regular and then its value is $1/n$.

5. The particular case of this theorem for a plane triangle ($n=2$) can be proved very easily by using the formula $OI^2 = R^2 - 2Rr$. This suggests an interesting question. Is it possible to obtain a similar formula (in terms of R and r alone) for the square of the distance between the incentre and the circumcentre of a tetrahedron (and of the general n-dimensional simplex)?

In Salmon's *Solid Geometry* Vol. I (sixth edition) p. 235 an expression is given for the square of the distance between these two points for a tetrahedron, but it involves the edges and the altitudes of the tetrahedron, and I was unable to derive from it another expression containing R and r alone. I would like to know if such a formula can be obtained for a tetrahedron (and a simplex), for it will provide the simplest proof of the main theorem of this note.

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P. K. KASHIKAR.

Remarks on the above paper

The late Prof. J. H. Grace has obtained the inequality given below in a paper entitled "Tetrahedra in relation to Spheres and Quadrics" published in *Proc. Lond. Math. Society* Vol. 17 (1918). Let r, R be the in- and circum-radii of a tetrahedron and d the distance between the in- and circum-centres.

Then

$$(R+r)(R-3r) \geqslant d^2$$

From the above it follows that, for a real tetrahedron, $r/R \leqslant 1/3$.

Annamalainagar. B. R. VENKATARAMAN.

GLEANINGS

When, however, with much effort I reached the thirteenth proposition of Euclid, the utter simplicity of the subject was suddenly revealed to me. A subject which only required a pure and simple use of one's reasoning powers could not be difficult. Ever since that time Geometry has been both easy and interesting for me.

M. K. GANDHI—*The Story of my experiments with Truth.*

"It became still more interesting when, after the angularities of a combination of straight lines, I learnt to portray the graces of a curve. How many properties were there of which the compasses knew nothing, how many cunning laws lay contained in embryo within an equation, the mysterious nut which must be artistically cracked to extract the rich kernel, the theorem! Take this or that term, place the + sign before it and forthwith you have the ellipse, the trajectory of the planets, with its two friendly foci, transmitting pairs of vectors whose sum is constant; substitute the − sign and you have the hyperbola with the antagonistic foci, the desperate curve that dives into space with infinite tentacles, approaching nearer and nearer to straight lines, the asymptotes, but never succeeding in meeting them. Suppress that term and you have the parabola, which vainly seeks in infinity its lost second focus; you have the trajectory of the bombshell; you have the path of certain comets which come one day to visit our sun and flee to depths whence they never return. Is it not wonderful thus to formulate the orbit of the worlds? I thought so then and I think so still."

FABRE,

(From L. G. SIMONS' *Fabre and Mathematics and other Essays.*)
Scripta Mathematica Library No. 4.

THE RULE OF SUCCESSION IN STATISTICAL INFERENCE

BY

S. SUBRAMANIAN AND S. P. SUBRAMANIAN

The problem of estimating the frequency of successes when the probability of success in a single event is given has been satisfactorily solved by Bernoulli, but the converse problem of obtaining a measure of probability from an observed statistical frequency remains unsolved even at the present day. Bernoulli did not proceed beyond suggesting a possible method by means of an inversion of his famous theorem. One has to come down to the time of Laplace to find a serious attempt at solving the problem. Laplace followed two methods of which the first assumes an inversion of Bernoulli's theorem and the second involves an application of Baye's Rule. His aim was to see if, when the frequency of successes in a given number of trials was known, the probability of success at subsequent trials could be inferred with accuracy. Laplace's second method has led to what is called* the Rule of Succession which says :—*If an event has occurred m times and failed n times under given conditions and we do not know anything more than that the probability of its occurrence when the conditions are next fulfilled is* $(m+1)/(m+n+2)$. There are many who do not accept the validity of Laplace's arguments. The lines along which the critics proceed are clearly indicated by Mr. Keynes in his *Treatise on Probability*. On the other hand there are those who agree with Laplace. Karl Pearson was among those who supported Laplace. Pearson restated Laplace's theorem and proved an extended form of his result by applying Bayes' Rule and making use of Gamma functions. It is not the purpose of this note to rake up the old controversy about the validity of the rule or otherwise but only to point out how the rule as extended by Pearson can be established from very simple considerations. In Part I of this paper a general theorem is proved and in Part II, Pearson's extension of Laplace's rule is derived as a particular case of the general theorem.

* Venn: *Logic of Chance*; p. 190.

PART I

1. Let us consider the following problem :—

A bag contains n balls of which any number may be white and the rest black ; P balls are drawn from the bag and of these p are found to be white. What is the probability that of R balls drawn next, r will be white?

Now, n may be finite or infinite and the drawing may be done in any one of three ways. The solution of the problem will therefore fall into various classes. Let $P=p+q$, $R=r+s$.

(1) *n is finite.*

(*a*) *The balls are drawn one by one but not replaced.*

The drawing of p white balls and q black balls is only one of $(p+q+1)$ equally likely cases starting from $(p+q)$ white balls and no black ball and ending with no white ball and $(p+q)$ black balls. Hence the probability of such a drawing is $1/(p+q+1)$.

Similarly the probability of drawing $(p+r)$ white balls and $(q+s)$ black balls *in any order* is $1/(p+q+r+s+1)$. The total number of balls in the two draws, namely, $p+q+r+s$ can be divided into groups of $(p+q)$ and $(r+s)$ in $(p+q+r+s)!/(p+q)!\ (r+s)!$ ways. Of these we want only those in which the $(p+r)$ white balls are divided into groups of p and r and the $(q+s)$ black ones into groups of q and s. There are only $\dfrac{(p+r)!}{p!\cdot r!}\cdot\dfrac{(q+s)!}{q!\,s!}$ such groups. Hence the probability that the white and the black balls have followed the given order is

$$\frac{(p+r)!\ (q+s)!\ (r+s)!\ (p+q)!}{p!\ q!\ r!\ s!\ (p+q+r+s)!}\cdot\frac{1}{p+q+r+s+1}=k \text{ (suppose).}$$

Thus if π were the probability that p white and q black ones will be followed by r white and s black, then since the trials are independent

$$\frac{1}{p+q+1}\cdot\pi=k\text{; that is,}$$

$$\pi=\frac{(p+r)!\ (q+s)!\ (r+s)!\ (p+q+1)!}{p!\ q!\ r!\ s!\ (p+q+r+s+1)!} \qquad \ldots \quad \text{(A)}$$

(*b*) *The P balls are drawn together at first and not replaced; then the R balls are drawn together*

The reasoning will be identical with that in the last subdivision and the answer will be the same.

(c) *The balls are drawn one by one and replaced after every attempt.*

In this case the problem becomes complicated and the argument of subdivision (a) is inapplicable; for, supposing that the first ball drawn is white and is replaced, before the second draw we know that there is at least one white ball in the bag whereas nothing is known about the black. Therefore we cannot say that all possible draws are equally likely.

(2) n is infinite

In cases (a) and (b) the arguments proceed as in section (1). But in case (c) it will be seen that the number of white balls may be infinite or the number of black ones or both and that in all these cases the fact that we draw a white ball and replace it does not give us more information about the white than about the black for the next drawing; the reason is that, here, the difference is between finite and infinite numbers. It may also be observed that the distinction between the cases of replacement and non-replacement arises only when the total number of balls is finite because only then the probabilities of drawing a white ball in the second trial will differ in the two cases. Thus result (A) holds good here also.

PART II

Karl Pearson's statement of Laplace's rule is as follows †:—

Let the chance of a given event occurring be supposed to lie between x and $x+\delta x$ and then if on $n(=p+q)$ trials the event has been observed to occur p times and fail q times........, the chance C_r, whatever x may be, of the event occurring r times in the second series of $m(=r+s)$ trials is given by

$$\frac{m!}{r!\,s!} \cdot \frac{\int_0^1 x^{p+r}(1-x)^{q+s}dx}{\int_0^1 x^p(1-x)^q\,dx}.$$

The method of the present paper and that of Laplace and Pearson can be related if we term the drawing of a white ball a success and that of a black one, a failure. The Laplace-Pearson method assumes that all values of x from 0 to 1 are equally likely and that x remains constant from ball to ball. In effect this is equivalent to

† Karl Pearson: *Philosophical Magazine* (1907) pp. 365-78.

assuming the number of balls in the bag to be infinite. This is case (2) of Part I of this paper. Pearson's formula for C_r can be easily verified to lead to result (A) of the paper on performing the integration.

The main difference between the Laplace-Pearson method and the one followed in this paper is in the distribution of ignorance. The former method would have it that every possible composition of the bag is equally likely whereas the latter assumes that every possible assortment of successes and failures in a given number of trials is equally likely. The latter method is more general than the former in that it provides for a limitation of the field of variation of the probability of success as also a change in the value of this probability from trial to trial. But it is unfortunate, however, that this general method should be inapplicable to the case where the probability of success may take any value within a limited field but is constant from trial to trial. Herein lies the weakness of all methods based on the principle of equal distribution of ignorance.

Conclusion

We have seen that a general rule of succession could be established by applying the principle of the equal distribution of ignorance in a manner different from that of Laplace and Pearson and that the Laplace-Pearson rule comes out as a particular case of the general rule formulated in this paper.

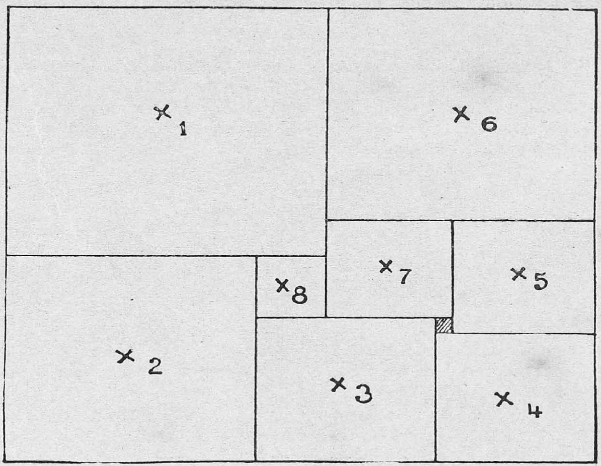
NOTES AND DISCUSSIONS

The division of a rectangle into unequal squares

§ 1. According to Steinhaus ("*Mathematical Snapshots*", published by G. E. Stechert & Co., New York) the problem of dividing a square into squares of *unequal* size, is still unsolved.

However it is possible to divide a rectangle into *nine* squares of unequal size. Steinhaus leaves the solution of this as an exercise to the reader; the end of his book contains a reference to a published solution, which is probably not available in India.

The following method seems to me a natural one for dividing a rectangle into 9 squares of unequal size. It is possible that a similar elementary method can be used to prove that it is *impossible* to divide a rectangle into *eight* unequal squares (an unsolved problem according to Steinhaus).



§ 2. Let us suppose that the lengths of the sides of the squares are x_n $(1 \leqslant n \leqslant 9)$, where no two x's are equal. Without loss of generality we may suppose that $x_9 = 1$.

The following diagram indicates a tentative division of a rectangle into squares of sides x_n $(1 \leqslant n \leqslant 9)$. In this figure all the rectangles (nine in number) are supposed to be squares of sides x_n. The shaded rectangle is a square of side x_9 $(=1)$.

If *this* division of a rectangle into unequal squares is possible then it must be possible to solve the following system of linear equations in the 8 unknowns $(x_1, x_2, x_3, \ldots, x_8)$ in *positive unequal* x's:

$$\begin{aligned} x_1 + x_2 &= x_4 + x_5 + x_6 \\ x_2 + x_3 + x_4 &= x_1 + x_6 \\ x_6 &= x_5 + x_7 \\ x_5 &= x_7 + 1 \\ x_1 &= x_2 + x_8 \\ x_3 &= x_4 + 1 \\ x_6 + x_7 &= x_1 + x_8 \\ x_8 + 1 &= x_7 + x_8. \end{aligned}$$

The reader will easily verify that this set of equations has the unique solution:

$$\begin{aligned} &x_1 = 18, \quad x_2 = 14, \quad x_3 = 10, \\ &x_4 = 9, \quad x_5 = 8, \quad x_6 = 15, \\ &x_7 = 7, \quad x_8 = 4. \end{aligned}$$

It follows that *the whole area of a rectangle of sides* 33 and 32, *can be divided into the nine unequal squares of sides*: 1, 4, 7, 8, 9, 10, 14, 15, 18.

S. CHOWLA.

On Stirling's Approximation

I. In this note, I give an easy proof for

Theorem: $\log (n!) = F(n) + C + \theta/3\,(n-1)$,

where $F(n) = n \log n - n + \frac{1}{2} \log n$; $|\theta| < 1$; C is a constant.

Let $G(n) = F(n) - F(n-1) - \log n$; and $K = \sum_{r=2}^{n} G(r)$.

Then $K = F(n) - F(1) - \log (n!) = F(n) + 1 - \log (n!)$. ... (1)

Further, $G(n+1) = n \log \left(1 + \frac{1}{n}\right) - 1 + \frac{1}{2} \log \left(1 + \frac{1}{n}\right)$. ... (2)

Since $\log (1+x) > x - \frac{1}{2} x^2$, from (2),

$$G(n+1) > \left(n + \frac{1}{2}\right)\left(\frac{1}{n} - \frac{1}{2n^2}\right) - 1 = -\frac{1}{4n^2}. \quad \ldots \quad (3)$$

Again, since $\log (1+x) < x - \frac{1}{2} x^2 + \frac{1}{3} x^3 < x$, from (2),

$$G(n+1) < n\left(\frac{1}{n} - \tfrac{1}{2}/n^2 + \tfrac{1}{3}/n^3\right) - 1 + \tfrac{1}{2}/n = \tfrac{1}{3}/n^2. \quad \ldots \quad (4)$$

From (3), (4), we get that K tends to a limit when $n\to\infty$,

$$\text{Let } A = \sum_{r=2}^{\infty} G(r).$$

Then, since $1/m^2 < 1/m\,(m-1) = \dfrac{1}{m-1} - \dfrac{1}{m}$,

$$|K - A| \leqslant \sum_{r=n+1}^{\infty} |G(r)| \leqslant \tfrac{1}{3} \sum 1/(r-1)^2 \leqslant \frac{1}{3\,(n-1)}.$$

So $K = A + \dfrac{\theta}{3\,(n-1)}$, where $|\theta| < 1$. ... (5)

From (1) and (5), we get the theorem, where $C = 1 - A$.

If we put $F(n) = n \log n - n + \frac{1}{2} \log n + \dfrac{1}{4n}$, similarly we can prove that $\log (n\,!) = n \log n - n + \frac{1}{2} \log n + \dfrac{1}{12n} + O\,(1/n^3)$.

II. By the above method, we can easily prove that

$$S(n) = \sum_{1}^{n} \frac{1}{r} = \log n + \gamma + \theta/n, \text{ where } 0 < \theta < 1.$$

Let $H(n) = \log n - \log (n-1) - 1/n$. ... (6)

Then $\sum_{2}^{n} H(r) = \log n - \sum_{2}^{n} 1/r = \log n + 1 - S(n)$. ... (7)

But, since $-\log (1-x) > x$, from (6),

$$H(n) = -\log (1 - 1/n) - 1/n > \frac{1}{n} - \frac{1}{n} = 0 \qquad \text{... (8)}$$

Again $H(n) = \log \left(1 + \dfrac{1}{n-1}\right) - \dfrac{1}{n} < \dfrac{1}{n-1} - \dfrac{1}{n}$. ... (9)

From (8), (9), we get that $\sum_{2}^{\infty} H(r)$ is convergent, and let

$$D = \sum_{2}^{\infty} H(r).$$

Then $0 < D - \sum_{2}^{n} H(r) = \sum_{n+1}^{\infty} H(r) < \sum_{n+1}^{\infty} \left\{\dfrac{1}{r-1} - \dfrac{1}{r}\right\} = \dfrac{1}{n}$.

So $\sum_{2}^{n} H(r) = D - \theta/n$, where $0 < \theta < 1$. ... (10)

From (7) and (10), we get that

$$S(n) = \log n + 1 - D + \theta/n.$$

Annamalainagar. S. S. PILLAI.

SOLUTIONS TO QUESTIONS

Question 1702

(K. SATYANARAYANA):—The common points of the circles on the diagonals of a quadrilateral as diameters are such that each side is the ortho-line of either of the points with respect to the triangle formed by the other three. Calling these *the ortho-points of the quadrilateral*, prove that:

(1) The circumcircle of the diagonal triangle and the self-polar circles of the four triangles of the quadrilateral form a coaxal system with the ortho-points as limit points.

(2) The director circles of all inconics of the quadrilateral form a coaxal system with the ortho-points as common points.

Hence prove Gaskin's theorem.

Show also that the necessary and sufficient condition for the existence of a rectangular hyperbola in the inconic family is the coincidence of the ortho-points.

Solution by K. Rangaswami

It is well known that the director circles of conics touching the sides of a quadrilateral belong to a coaxal system Γ to which also belong the circles described on the three diagonals as diameters. Thus the director circles have in common the two ortho-points of the quadrilateral.

Now, the conics outpolar to the range is an ∞^3-system which must contain ∞^1 circles i.e., a coaxal system Γ'. Since the ortho-points P, Q lie on the director circles of all conics of the range, the tangents from P and Q to the conics are at right angles. Thus, P and Q are point circles outpolar to the range and are, therefore, the limit points of the coaxal system Γ'.

Again, the circumcircle of the diagonal line triangle of the quadrilateral and the polar circles of the four triangles formed by the lines of the quadrilateral are evidently circles outpolar to the conics of the range. Hence, these circles belong to the coaxal sytem Γ' for which P and Q are limit points and cut orthogonally the

circles of the system Γ. We have, therefore, Gaskin's theorem that any circle outpolar to a conic cuts orthogonally the director circle of the conic.

Now, among the conics of the range there will be two rectangular hyperbolas and their director circles are the point circles at their centres, say, R, R'. Thus R, R' are the limit points of the coaxal system Γ. If, however, there is only one rectangular hyperbola in the range, R and R' must coincide and Γ is a touching coaxal system. Hence their common points P, Q *viz.*, the ortho-points of the quadrilateral must also coincide.

Also solved by T. K. Raghavachari and M. V. Vaidyanatha Sastry.

Question 1745

(KAPREKAR D. R.) :—We have $91 \times 819 = 74529$ and also $9901 \times 980199 = 9704950299$ where the corresponding numbers in the second multiplication are obtained by inserting 9 and 0 alternately before the first digit and between the successive digits of all the numbers in the first multiplication. It will be found that the multiplication is still true if we insert alternately blocks of 99 and 00 or 999 and 000 and so on. It is required to find the principle underlying the result and to find other number pairs like 91 and 819 with the same property.

Further Remarks by V. Narasimhamurthi.

A discussion of this was published by me in Vol. VI of *The Mathematics Student* pp. 83-85. The proposer remarks on p. 85 that he has not succeeded in getting any general formula for obtaining number-pairs with the property stated in the question. The number pairs, 81, 729; 71, 819; 81, 819; 92, 729; which were stated by him, satisfy the limitations of my general formula of p. 84. Hence these pairs satisfy the required property.

Mr. Kaprekar gives also two number pairs, wherein we have to regard groups of two digits as unit between which groups of 9's and 0's are to be inserted. They are

$$\left.\begin{array}{l} 940392 \times 9302 = 87\ 47\ 52\ 63\ 84 \\ 994003992 \times 993002 = 9870479\ 52063984 \end{array}\right\} \quad \text{(i)}$$

$$\left.\begin{array}{l} 970193 \times 980192 = 95\ 09\ 75\ 41\ \cdot 7056 \\ 997001993 \times 998001992 = 995009975041970056 \end{array}\right\} \quad \text{(ii)}$$

We may also insert r nines and r zeros in the gaps instead of single nines and single zeros. My solution was based on the relation

$$[x^m - a_1x^{m-1} + (a_2+1)x^{m-2} \mp \cdot \cdot \cdot \cdot] \times [x^n - b_1x^{n-1} + (b_2+1)x^{n-2} \mp \cdot \cdot \cdot \cdot \cdot]$$
$$\equiv [x^l - c_1x^{l-1} + (c_2+1)x^{l-2} \mp \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot]$$

For this above property to hold, this identity must be satisfied for every $x = 10^t$ where $t > 1$. In that case the property in the question fails and the property in the remark can be obtained. But the upper limits of c's will be different in this case.

$$10 > a_1,\ b_1 > 0;\ 10 > a_{2r},\ b_{2r} \geqslant 0;$$
$$10 \geqslant a_{2r+1},\ b_{2r+1} > 0;$$

And on account of t being greater than unity, at least one of the c's exceeds 10; and all the c's are less than 10^2, (because the identity is satisfied for $t=2$.) In the example (i),

$$a_1 = 6,\quad a_2 = 3,\quad a_3 = 8;$$
$$b_1 = 7,\quad b_2 = 2;$$

The identity of my solution gives

$$c_1 = 13,\ c_2 = 48,\ c_3 = 47,\ c_4 = 63,\ c_5 = 16;$$

Here the c's are all greater than 10, and are all less than 10^2.

In the example (ii) also,

$$a_1 = 3,\ a_2 = 1,\ a_3 = 7;\ b_1 = 2,\ b_2 = 1,\ b_3 = 8;$$
$$c_1 = 5,\ c_2 = 9,\ c_3 = 25,\ c_4 = 41,\ c_5 = 30,\ c_6 = 56;$$

Some of the c's exceed 10, and all are less than 10^2.

We can obtain as many number pairs satisfying this second property as we wish

$$910198 \times 9302 = 84\ 66\ 66\ 17\ 96;$$
$$9501 \times 9401 = 89\ 31\ 89\ 01;$$

and so on.

On the other hand, if the identity of my solution does not hold for $x = 10^2$ this second property will be violated. And in case it holds for $x = 10^3$, we get a third property *viz.* of regarding groups of three digits as units between which groups of 9's and 0's are to be inserted. Examples:

$$\left.\begin{array}{l} 991009 \times 991009 = 932\ 098\ 838\ 081 \\ 99910009 \times 99910009 = 99320098983800081 \end{array}\right\} \quad \text{(i)}$$

$$991009992001 \times 991009993001 = 982\ 100\ 805\ 236\ 832\ 075\ 98500,$$

$$\left.\begin{array}{l} 9991000999920001 \times 9991000999930001 \\ \quad = 998201009805023698320075998500001 \end{array}\right\} \quad \text{(ii)}$$

In the first of these two examples,

$$a_1 = 9, \quad a_2 = 9;$$
$$b_1 = 9, \quad b_2 = 9;$$

$c_1 = 18$, $c_2 = 98$, $c_3 = 162$, $c_4 = 81$; Here at least one of the c's exceeded 10^2 and all the c's are less than 10^3.

Therefore we generalise that, when the following conditions hold, we have the property of regarding groups of t digits as units between which groups of nines and zeroes are to be inserted:

Conditions:

(1) $10^t > c_{2r}$, $10^t \geqslant c_{2r+1}$, $t \geqslant 2$;

This condition must hold for all the c's.

(2) $c_{2r} \geqslant 10^{t-1}$, $c_{2r+1} > 10^{t-1}$, $t \geqslant 2$;

This condition must hold for at least one of the c's.

We may also mention that any number-pair, having in each number more than ten digits, cannot have the first property since some of the c's exceed the limit 10 whatever be the values of the digits a and b.

Question 1765

D. R. KAPREKAR :—If A, B, C, D are four integers each containing the same number of digits such that $(A + C) = (9)^k$ and $(B + D) = (9)^k$ then

$$(A \times B) + (C \times D) + (A \times D) + (B \times C) + A + B + C + D = (9)^{2k}$$

(The symbol $(9)^l$ is the integer formed by the digit 9 repeated l times).

Solution by V. Narasimhamurti, V. Krishnan, B. R. Venkataraman, J. Mahrenholz, B. G. Pendse, H. Z. Gilmani, G. Venkatasubbiah, and the proposer:

$$\text{Now } (A \times B) + (C \times D) + (A \times D) + (B \times C) + A + B + C + D$$
$$= (A + C)(B + D) + (A + C) + (B + D)$$
$$= (A + C + 1)(B + D + 1) - 1 = 10^{2k} - 1 = (9)^{2k}$$

Further Remarks by B. G. Pendse

It should be noticed that the proof is not affected even if we change A, B, C, D provided A + C and B + D remain the same. Hence the restriction that A, B, C, D should have the same number of

digits may be omitted. Thus let $A=8$, $C=91$; $B=2$, $D=97$ so that $A+C=B+D=99$. We still have

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+A+B+C+D=9999$$

The following results are generalisations of Question 1765:

(1) If A, B, C, D are integers such that

$$(A+C)=(9)^l,\ (B+D)=(9)^m \text{ then}$$

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+(A+B+C+D)=(9)^{l+m}$$

(2) If A, B, C, D are integers such that

$$(A+C)=(p)^l \text{ and } (B+D)=(p)^m$$

p being any integer of one digit, then

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+(A+B+C+D)\frac{p}{9}=\frac{p}{9}\ (p)^{l+m}$$

when $p=9$ this reduces to (1)

(3) If A, B, C, D are integers such that $(A+C)=(p)^l$ and $(B+D)=(q)^m$, p and q being integers with a single digit then

$$(A\times B)+(C\times D)+(A\times D)+(B\times C)+\frac{q}{9}(A+C)+\frac{p}{9}(B+D)=\frac{pq}{9}\times(1)^{l+m}$$

To prove the last result, we note that the left side is

$$(A+C)(B+D)+\frac{q}{9}(A+C)+\frac{p}{9}(B+D)$$

$$=(p)^l(q)^m+\frac{q}{9}(p)^l+\frac{p}{9}(q)^l$$

$$=\frac{pq}{81}\left\{(10^l-1)(10^m-1)\right\}+\frac{pq}{81}\left\{10^l-1+10^m-1\right\}$$

$$=\frac{pq}{81}(10^{l+m}-1)=\frac{pq}{9}(1)^{l+m}$$

Generalisation by the Proposer

If $(A+C)=(p)^k$, $(B+D)=(p)^k$ placing a single digit number then $(A\times B)+(C\times D)+(A\times D)+(B\times C)+(A+B+C+D)+(p)^k\ (9-p)^k=(p)^{2k}$.

Proof: The left side may be written

$$(A+C)(B+D)+(A+B+C+D)+(p)^k(9-p)^k$$

$$=(p)^k(p)^k+2(p)^k+(p)^k(9-p)^k$$

$$=(p)^k\left\{p(1)^k+2+(9-p)(1)^k\right\}=p\frac{(10^k-1)}{9}\times(10^k+1)$$

$$=p\frac{(10^{2k}-1)}{9}=(p)^{2k}.$$

REVIEWS

L. V. GURJAR : *Theory of Elasticity*, Arya Bhushan Press, pp 108, 1937, Price Rs. Four.

This small brochure of about a hundred pages is devoted mainly to the derivation of the principal differential equations, and other fundamental relations of the Theory of Elasticity, and is based on the lectures delivered by the author to post-graduate students. On the *mathematical* side the treatment appears to be unnecessarily long. If the author had used the notations of the vector and tensor calculus the subject-matter of the book could have been condensed to half the original volume, and the value of the book would have been undoubtedly enhanced. On the *physical* side the book does not go very deep either. It is one thing to give the derivation of the differential equations of the problem, but quite another thing to solve these equations subject to given initial and boundary conditions. About this latter fundamental point there is no reference whatsoever. At least the characteristics of boundary value problems in so far as they relate to Elasticity might have been briefly given. Again there is no mention made of the Theory of Plasticity which is a recent branch of Elasticity fundamentally modifying some of the general concepts of the theory.

We also find it hard to agree with the author that a study of the theory of Elasticity is of *vital significance* to the development of Theoretical Physics, which appears to us to be an over-statement of facts.

In the restricted domain to which the author confines himself he has written a clear and readable book which will surely be useful to post-graduate students.

B. S. MADHAVA RAO

K. R. GUNJIKAR, *An introduction to the calculus. Part I Differential Calculus*, Oxford University Press, pp. 182.

Text-books on calculus usually fall into two distinct types. The first type is meant as a first introduction of the subject to somewhat immature students, Graphs and geometric imagery are freely used and full advantage is taken of geometric intuition. The second type is intended for advanced students, and aims at a rigorous treatment of the fundamentals of the subject. There is noticeable in the first type of books a tendency to indulge in loose statements, which may have to be unlearnt at a later stage, and to shift the emphasis from a grasp of the fundamental principles to the attainment of mechanical proficiency in solving problems. In the other type, there is the danger that an undue insistence on rigour to the exclusion of whatever intuition has to tell us, and a disdain of the many practical applications may choke the enthusiasm of the learner, by making the subject a dry and highly introverted science. For the B. Sc. students of Indian universities, neither of these types is quite suited. There is need for striking out a *via media* plan, in which

beginning from ideas suggested by intuition, one is led by gradual steps to a rigorous analytic formulation, and there is a clear statement of what are assumed and what are proved. Professor Gunjikar has brought to bear his long experience in the teaching of calculus, in getting up a book supplying this need.

The limit idea, which is fundamental in calculus is first taken up. The intuitive concepts of the speed of a particle, and the tangent to a curve are analysed, and the way is prepared for the introduction of the differential coefficient. Continuous functions and the various types of discontinuity are studied in chapter III. Infinitesimals, and differentials are carefully treated. Chapter VII is devoted to real numbers, Dedekindian sections and real variable. The exponential and logarithmic functions are introduced, by assuming that the curve $y=10^x$ has a gradient at the point (0, 1). It is refreshing to find the historical note at the end of chapter IV lending a human interest to the subject.

Considerable pains have been taken to expound the theory in a lucid and logical manner, and point out its applications by means of graded and well-chosen sets of examples.

The get-up-excellent. In § 3·21, the symbol l may easily be mistaken for *1*, and the difficulty may be avoided by keeping to small *l*.

It is stated that the text-book is meant as a first course for College students. Rightly enough, the elements of calculus are introduced in the Intermediate classes of most of the Indian universities. It is deplorable that there is no uniformity in the syllabus. Still the book under review seems to be too advanced for the Intermediate for the freshmen if at all it is meant for them. It can be strongly recommended as an excellent text-book for the B. Sc. students of Indian universities. A chapter or at least an article on curvature might have been, with profit, incorporated.

B. RAMAMURTI.

M. BIERNACKI: *Les Fonctions Multivalentes,* (Actualités Scientifiques et Industrielles, No. 657). Price 20 frcs.

This booklet aims at bringing together the results of the recent researches on the Theory of Multivalent Functions. The notion of multivalence is due to P. Mortel who was the first to study systematically the properties of such functions. A function $f(z)$ is said to be *p-valent* in a domain D if it does not take *more than p times* any of its values in D *and* takes *exactly p times* at least one of its values in D. When $f(z)$ takes every one of its values exactly p-times in D it is said to be exactly p-valent. Some allied notions are also considered in the booklet under review, especially that of quasi-multivalence which restricts p-valence to a set of values belonging to a given aggregate, the remaining values being taken more than p-times in D.

When $p=1$, we get univalent functions. These are otherwise known as simple (*schlicht*) functions. When $f(z)$ is regular and simple in a domain D the relation $w=f(z)$ represents D on another domain in a conformal and one-to-one manner. The properties of functions simple in $|z|<1$ has

been studied in great detail. The researches on multivalent functions have been mainly directed to prove and generalise results already known for simple functions. For instance, if $f(z)=a_1 z+a_2 z^2+\cdots$ be simple in $|z|<1$, then $a_n=O(n)$ and $f(z)=O[(1-r)^{-2}]$ as $r=|z|\to 1$. Miss Cartwright has proved that if $f(z)=a_p z^p+a_{p+1}z^{p+1}+\cdots$ be p-valent in $|z|<1$, then $f(z)=O[(1-r)^{-2p}]$ as $r\to 1$. Similarly it has been proved that $a_n=O(n^{2p-1})$ and these are the best possible results.

The Introductory chapter in the booklet under review contains the definitions of p-valence and allied notions, some conditions for p-valence and an account of the relation between quasi-normal families and p-valent analytic functions. Chapter I is devoted to a study of multivalent holomorphic functions, Chapter II to meromorphic functions and the last chapter contains some properties of sets of functions in relation to their multivalence.

There is an exhaustive bibliography at the end. As the author himself remarks, the booklet is "intended to collect and group together the results on Multivalent Functions so as to facilitate further researches on the subject."

V. GANAPATHY IYER.

QUESTIONS FOR SOLUTION

1775. (A. A. KRISHNASWAMI AYYANGAR):—Prove the following results:

$$\frac{\pi}{\sqrt{12}}=1-\frac{1}{3\cdot 3}+\frac{1}{5\cdot 3^2}-\frac{1}{7\cdot 3^3}+\cdots$$

$$\frac{\pi}{16}=\frac{1}{1^5+4\cdot 1}-\frac{1}{3^5+4\cdot 3}+\frac{1}{5^5+4\cdot 5}-\cdots$$

$$\frac{\pi-3}{4}=\frac{1}{3^3-3}-\frac{1}{5^3-5}+\frac{1}{7^3-7}-+\cdots$$

$$\frac{4-\pi}{8}=\frac{1}{4^2-1}+\frac{1}{8^2-1}+\frac{1}{12^2-1}+\cdots \quad (\textit{Tantrasangraha})$$

$$\frac{\pi-3}{6}=\frac{1}{(2\cdot 2^2-1)^2-2^2}+\frac{1}{(2\cdot 4^2-1)^2-4^2}+\frac{1}{(2\cdot 6^2-1)^2-6^2}+\cdots$$

(*Karanapaddhati*, 1730)

[The invention of infinite series of the above forms orginated in Malabar as early as the 15th century. In the *Transactions of the Royal Asiatic Society of Great Britain and Ireland* Vol. III Part III (1834), Charles M. Whish of the Hon. East India Company's Civil Service in the Madras Establishment draws our attention to these series and writes: "The author of *Karanapaddhati* whose grandson is now alive in the 70th year was Pathumana Somayaji, a Nambudri Brahmin of Trichur in Malabar. . . . . The Author of *Tantrasangraha* who was educated in the College of Trichur laid the

foundations of a complete system of fluxions. His name is Tala Kalattura Nambudri of Kerala and he flourished in the 46th Century of *Kaliyuga* (4532 *Kali*—1432 A. D.) . . . . The first series for $\pi/\sqrt{12}$ is exactly similar to that laid down by Dr. Halley and communicated to the *Royal Society of London*."

Will any historian of Malabar give any further details about these authors and their works?]

1776. (B. R. VENKATARAMAN):—Show that the envelope of lines in a plane π which are equidistant from two fixed points A, B (not lying on π) is a parabola; and that as π revolves about a fixed line l on it, the focus of the parabola describes a circle on PM as diameter in a plane normal to l, where P is the mid-point of AB and M is the foot of the perpendicular from P on l.

1777. (V. THEBAULT, Le Mans, France):—Find a square of three digits abc given that the number of 19 digits $a99999999b00000000c$ is also a perfect square.

1778. (D. R. KAPREKAR):—A number $abcxylmn$ of eight digits divided by a number N of six digits yields a quotient equal to the integral part of $(abc)/3$ with a remainder $1xy999$.

Example: $13313867 \div 299997 = 44 + (113999 \div 299997)$.

Prove that there are only 300 numbers $abcxylmn$ and find the greatest and least of them.

1779. (S. CHOWLA) Is it possible to divide the volume of a rectangular parallelopiped into *unequal* cubes?

(It is known that the area of a rectangle can be divided into nine unequal squares.)

19025

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